Group Theory, Topology, and Physics

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Group Theory

1. A *Group* is a set G with a binary operation (denoted just as ordinary multiplication) satisfying three axioms:

   (I) The operation is associative: $g(hk) = (gh)k$ for all $g, h, k \in G$.

   (II) There exists an identity $1_G \in G$ such that $(\forall\, g)\ 1_G g = g 1_G = g$.

   (III) $(\forall\, g \in G)\ \exists\, g^{-1} \in G$ such that $gg^{-1} = g^{-1}g = 1_G$.

2. Example: let $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ with addition modulo n. This is clearly a group, but what about multiplication? One can easily see that only elements of $\mathbb{Z}_n$ that are relatively prime to n have multiplicative inverses. Therefore, $\mathbb{Z}_n$ will be a group under multiplication iff n is prime. Even if n is not prime, the set of invertible elements in $\mathbb{Z}_n$ is a group.

3. If A, B are two sets, the direct product $A \times B$ is defined to be

   $$A \times B = \{(a, b) : a \in A, b \in B\}.$$

   If A and B also happen to be groups, then the set $A \times B$ has a natural group structure defined by performing all operations componentwise. The identity of $A \times B$ is, of course $(1_A, 1_B)$.

4. Let G, H be groups. A map $\phi : G \to H$ is called a *homomorphism* if $\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$. **Exercise:** It follows from this that $\phi(1_G) = 1_H$ and $\phi(g^{-1}) = \phi(g)^{-1}$, so it is not necessary to assume these as additional properties. A homomorphism is said to be an *isomorphism* if it's 1-1 and onto.

5. Example: let R_n denote the set of nth roots of unity, i.e. the complex roots of the polynomial $x^n - 1$. They form a regular n-gon on the unit circle including the point 1, and they are a group under multiplication. Let $z = \exp(2\pi i/n)$ be the fundamental root. Then $\phi : \mathbb{Z}_n \to R_n$ defined by $\phi(m) = z^m$ turns out to be an isomorphism.

6. A **subgroup** is a subset $H \subseteq G$ which is also closed under multiplication and inverses, and contains the identity. In other words, H is also a group with the same operation. A **left coset** of H is a set of the form

$$gH = \{gh \mid h \in H\}\,.$$

Theorem 1. *Two left cosets g_1H and g_2H are the same if and only if $g_1^{-1}g_2 \in H$.*

Proof. Suppose $g_1H = g_2H$. Then $\forall h \in H, \exists h_1 \in H$ such that $g_1h = g_2h_1$. Then $g_2^{-1}g_1h = h_1$ which proves that $g_2^{-1}g_1H \subseteq H$. By the cancellation law, this can only happen if $g_1^{-1}g_2 \in H$. Now for the converse, suppose $g_1^{-1}g_2 = h \in H$. Then $g_2 = g_1h$ hence $g_2H = g_1hH = g_1H$. $\square$

7. Let G/H denote the set of left cosets of H in G, and investigate when the operation

$$(g_1H) * (g_2H) = (g_1g_2)H \tag{1}$$

is well-defined. This is easy given Theorem 1. Suppose we chose different representatives for the two cosets above: $g_1'H$ and $g_2'H$ instead of g_1H and g_2H. Then $g_1^{-1}g_1' \in H$ and $g_2^{-1}g_2' \in H$. In order that the product $(g_1'H) * (g_2'H)$ yield the same answer, we must have $(g_1g_2)^{-1}(g_1'g_2') \in H$. But

$$(g_1g_2)^{-1}(g_1'g_2') = g_2^{-1}g_1^{-1}g_1'g_2' = g_2^{-1}hg_2' \tag{2}$$

where we define $h = g_1^{-1}g_1' \in H$. We also know that $g_2' = g_2h_1$ for some $h_1 \in H$. So we can rewrite (2) as $(g_1g_2)^{-1}(g_1'g_2') = g_2^{-1}hg_2h_1$, and the latter is in H if and only if

$$g_2^{-1}hg_2 \in H\,.$$

We are led to the inevitable conclusion that if H is a special kind of subgroup, for which $g_2^{-1}hg_2 \in H$ whenever $h \in H$ (such a subgroup is said to be **normal**) then (1) is well-defined.

Limits and Continuity

8. Let X and Y be spaces with distance functions d_X and d_Y respectively. (For example, the torus inherits a distance function from an embedding in $\mathbb{R}^3$). A map $f : X \to Y$ is said to be *continuous* if, given $\epsilon > 0$, there exists $\delta > 0$ such that $d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \epsilon$.

9. We give a second, equivalent definition of continuity. Let $(a_n), n \in \mathbb{N}$ be a sequence of points in a space X. We write $a = \lim_{n\to\infty} a_n$ in the case that $d_X(a_n, a) \to 0$ as $n \to \infty$. A map $f : X \to Y$ is *continuous* if $f(a) = \lim f(a_n)$ whenever $a = \lim a_n$. One might say that f "commutes with" the limit operation.

Loops and Homotopy

10. A *based space* is an ordered pair (X, x_0) where X is a topological space, and $x_0 \in X$ is a point.

11. A space X is said to be *(path) connected* if, given any two points $p, q \in X$, there exists a continuous map $\gamma : [0,1] \to X$ with $\gamma(0) = p$, and $\gamma(1) = q$.

12. A *loop* in a based space (X, x_0) is a continuous map $\alpha : [a,b] \to X$, such that $\alpha(a) = \alpha(b) = x_0$. Here, $[a,b]$ is any closed interval. Note the following important re-parameterization invariance: if $\gamma : \mathbb{R} \to \mathbb{R}$ is a continuous real function, which maps some other interval $[c,d]$ onto $[a,b]$, then $\alpha \circ \gamma : [c,d] \to X$ is also a loop which has the same image in X as the original loop α. We will not distinguish between loops which are related to one another by re-parameterization.

13. Let α and β both be defined on the interval $[a,b]$. A *homotopy* of two loops α, β is a continuous map $F : [a,b] \times [0,1] \to X$ such that $F(s,0) = \alpha(s)$ and $F(s,1) = \beta(s)$. One also sometimes writes $F_t(s) = F(s,t)$ so that each F_t is a loop. Thus a homotopy is a continuous path in the space of loops.

14. An *equivalence relation*, denoted $\sim$, is a relation that is reflexive ($f \sim f$), symmetric ($f \sim g \Rightarrow g \sim f$), and transitive ($f \sim g$ & $g \sim h \Rightarrow f \sim h$). Homotopy of loops is an equivalence relation. Let $[\alpha]$ denote the class of all loops homotopic to α.

The Fundamental Group

15. Given a based space (X, x_0), the *fundamental group* is denoted $\pi_1(X, x_0)$. It is the set of homotopy classes of loops based at x_0, with a group operation that we now describe. Let α and β be loops from $[0,1] \to X$. Define a new loop $\alpha \cdot \beta$ by

$$(\alpha \cdot \beta)(s) = \begin{cases} \alpha(2t) & t \in [0, \frac{1}{2}] \\ \beta(2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$$

    We then define the product of two loop classes $[\alpha] \cdot [\beta] = [\alpha \cdot \beta]$. One must check that this is operation is well-defined on equivalence classes, i.e. that $[\alpha \cdot \beta]$ only depends upon α and β through their homotopy class. This is the promised group operation for $\pi_1(X, x_0)$.

16. Let $f : X \to Y$ be continuous. Note that if α is a loop in X then $f \circ \alpha : [0,1] \to Y$ is a loop in Y. Define

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$$

by the formula $f_*([\alpha]) = [f \circ \alpha]$. One can check that f_* is well-defined on equivalence classes of loops, and is (in fact) a homomorphism, although it need be neither surjective nor injective in general. Let us check that f_* is a homomorphism.

$$f_*([\alpha \cdot \beta]) = [f \circ (\alpha \cdot \beta)] \tag{3}$$

while

$$f_*([\alpha]) \cdot f_*([\beta]) = [(f \circ \alpha) \cdot (f \circ \beta)] \tag{4}$$

However $f \circ (\alpha \cdot \beta) = (f \circ \alpha) \cdot (f \circ \beta)$ up to a possible re-parameterization of the intervals used to define the loops. Therefore, the two expressions (3) and (4) are equal.

17. A connected space X is said to be *simply connected* if $\pi_1(X, x_0) = \{\mathrm{id}\}$ for some (hence all) $x_0 \in X$.

18. A special case of van Kampen's theorem. Let $X = U \cup V$, where U and V are open sets with $U \cap V$ connected and simply connected. Then $\pi_1(X) \cong \pi_1(U) * \pi_1(V)$, where $*$ denotes the free product.