MAKING THE REALS FROM THE RATIONALS, WEEK 1

MATTHEW TAI

We ask, given the rational numbers, how do we get from there to the real numbers? Can we get there by taking square roots? This gives us some real numbers, but not all of them.

We took a look at Cantor's Diagonal argument for a moment. After making sure that everyone in the class knew what the argument meant, I asked if we could represent the real numbers as sequences of fractions, rather than as decimal expansions, and more specifically, how long must those sequences be if we were to be able to represent all real numbers.

Based on analogy with Cantor's Diagonal argument, we decided that the sequences would have to be infinitely long.

Next I asked what the sequence $(0, 0, 0, 0, ...)$ would be. We decided that this sequence would be 0.

Then I asked what the sequence $(1, 1, 1, 1, ...)$ would be. After some debate, we decided that this sequence would be 1.

What about $\frac{1}{2}$? We came across three sequences that could be $\frac{1}{2}$:

$$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, ...\right)$$

$$(1, 0, 1, 0, ...)$$

$$(0, 1, 0, 1, ...)$$

We chose the latter two because the average of 1 and 0 is $\frac{1}{2}$.

Then I asked the students what would happen if we decided that the sequence must get "arbitrarily close" to whatever number it is supposed to represent. Then the two sequences $(1, 0, 1, 0, ...)$ and $(0, 1, 0, 1, ...)$ wouldn't be able to represent $\frac{1}{2}$.

I asked then what $(1, 2, 3, 4, ...)$ would be. The decision was that this sequence would be ∞. But we eventually decided that ∞ is not a real number.

Then I asked about the sequence $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...)$. We decided that since this sequence got arbitrarily close to 0, that it must be 0.

At this point, asking for a definition of arbitrarily close returned that given any interval around the target number, there was an element of the sequence in that interval.

So I asked about the sequence $(1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, ...)$. For any interval around 0, there was an element of this sequence in that interval, but similarly for 1. Since we wanted each sequence to only represent one number, we decided that this sequence couldn't represent 0 or 1, and thus that our definition for arbitrarily close needed to be fixed.