

THE FOUR NUMBERS GAME, WEEK 1

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We start the game by drawing a square, and placing a nonnegative whole number at each corner.

Now we consider a side and the two adjacent corners. We take the (positive) difference between the numbers on those corners and place the difference in the middle of the side.

We do this for all four sides, and then draw a new square (tilted at 45°). We consider this to be one "step".

Note that if all four numbers on the corners are 0, then after one step, we again have 0 on all four corners. We say that if this happens then the game has terminated.

The goal of the game is to find four numbers such that it takes as many steps as possible for the game to terminate. We call the number of steps it takes for a game to terminate the "length" of the game.

For convenience, we'll write a state of the game as just a four-tuplet of number (a, b, c, d) , so that the next step would be $(|a - b|, |b - c|, |c - d|, |a - d|)$, where $|x|$ is the absolute value of x (i.e. x if x is positive, and $-x$ if x is negative)

For example, we considered a game starting with $(8, 4, 2, 1)$.

We got:

$(8, 4, 2, 1)$

$(4, 2, 1, 7)$

$(2, 1, 6, 3)$

$(1, 5, 3, 1)$

$(4, 2, 2, 0)$

$(2, 0, 2, 4)$

$(2, 2, 2, 2)$

$(0, 0, 0, 0)$

Thus the game $(8, 4, 2, 0)$ has length 7.

First we ask if all games terminate.

We observed that if all the numbers in a four-tuplet are greater than 0, then the largest number in the four-tuplet is greater than the largest number in the four-tuplet after taking one step. For example, the largest number in $(8, 4, 2, 1)$ is 8, while the largest number in the next step $(4, 2, 1, 7)$ is 7.

Hence we must eventually get a 0.

Through experimentation we showed that if there is a 0, then eventually the game will end.

Having shown that all games end, we decided to take a look at what happens when the numbers aren't positive whole numbers.

If we let the numbers be negative whole numbers, then after taking one step, all the entries are positive, so games with negative entries terminate.

From here, we noticed that if you add a constant to all of the entries, the next step doesn't change.

What about games with fractional entries? We noticed that if you multiply all of the entries in a tuple by a non-zero constant C , then taking a step yields the step after the original tuple multiplied by that non-zero constant C .

For example, if we take $(8, 4, 2, 1)$ and multiply it by 2, we get $(16, 8, 4, 2)$, and one step after that is $(8, 4, 2, 14)$, which is just $(4, 2, 1, 7)$ multiplied by 2. Therefore multiplication by a constant doesn't change the number of steps it takes for the game to terminate.

Thus if we multiply the fractions by all of the denominators, we get a game with whole numbers, which must terminate; therefore games with fractions all terminate.

Combining the fact that adding doesn't change the number of steps, and the fact that multiplying doesn't change the number of steps, we say that two games are "equivalent" if we can get from one to the other by adding constants to all the entries and by multiplying all the entries by constants.

We ended with the question of whether we could undo a step, i.e. given four numbers a, b, c and d , are there four numbers e, f, g and h such that taking (e, f, g, h) and taking a step would yield (a, b, c, d) .